

Gravitation theory in Riemann-Cartan space-time and regular cosmology

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Abstract. Principal ideas of gauge approach applying to gravitational interaction and leading to gravitation theory in Riemann-Cartan space-time are discussed. The principal relations of isotropic cosmology built in the framework of the Poincare gauge theory of gravity and the most important their consequences are presented.

1 Introduction

As it is well known, methods of Non-Euclidian Geometry are widely applied in modern theory of gravitation. According to Einsteinian General Relativity theory (GR) physical space-time has the structure of 4-dimensional pseudo-riemannian continuum. GR is the base of modern theory of gravitation and permits to describe successfully different gravitating systems with widely changing values of physical parameters (energy density, pressure etc). At the same time, GR leads to non satisfactory consequences in the case of gravitating systems at extreme conditions with extremely high energy densities, pressures, temperatures, where singular states with divergent values of certain physical parameters appear. Corresponding problem of GR – the problem of gravitational singularities – was widely discussed in literature [1]. One of the most important cases of this problem is the problem of cosmological singularity (PCS). All Friedmannian homogeneous isotropic cosmological models (HICM) of flat and open type and the most part of closed models possess singular state in the beginning of cosmological expansion, where the value of the scale factor $R(t)$ of Robertson-Walker metrics vanishes, that limits their existence in the past. According to wide known opinion, the solution of PCS and generally of the problem of gravitational singularities of GR has to be connected with quantum gravitational effects, which must be essential at Planckian conditions, when energy density is comparable with the Planckian one. Previously some regular bouncing cosmological solutions were obtained in the frame of candidates to quantum gravitation theory – string theory/M-theory and loop quantum gravity (see, for example, [2–4]). From physical point of view, these solutions have some difficulties [5]. In the case of bouncing cosmological solutions built in the frame of string theory the condition of energy density positivity for gravitating matter is violated. In the case of loop quantum cosmology a bounce takes place for microscopic model having a volume comparable with the Planckian one. If one supposes that the Universe at compression stage is macroscopic object, one has to explain the transformation of macro-universe into micro-universe before a bounce. This

means one has to introduce some physically non realistic model, which has to be inverse with respect to inflation.

As it was shown in a number of our papers (see [5-7] and Refs herein), gravitation theory in 4-dimensional physical Riemann-Cartan space-time leads to the solution of PCS and permits to build satisfactory regular Big Bang theory. Regular character of all solutions for HICM in this theory is ensured by gravitational repulsion effect at extreme conditions, when the energy density and the pressure of gravitating matter are extremely high (although their values can be essentially less, than the Planckian ones). The principal role at extreme conditions in such theory plays the space-time torsion. The use of Riemann-Cartan geometry in gravitation theory is motivated by applying the local gauge invariance principle in theory of gravitational interaction. The present paper is devoted to discussion of these problems. In Section 2 some principal ideas of gauge approach in gravitation theory are presented. In Section 3 the most important results of regular cosmology built in 4-dimensional Riemann-Cartan space-time are given.

2 Gauge approach to gravitational interaction and gravitation theory in Riemann-Cartan space-time

As it is known, the local gauge invariance principle is the basis of modern theory of fundamental physical interactions. The theory of electro-weak interaction, quantum chromodynamics, Grand Unified models of particle physics were built by using this principle. From physical point of view, the local gauge invariance principle establishes the correspondence between certain important conserving physical quantities, connected according to the Noether's theorem with some symmetries groups, and fundamental physical fields, which have as a source corresponding physical quantities and play the role of carriers of fundamental physical interactions. The applying of this principle to gravitational interaction leads, generally speaking, to generalization of Einsteinian theory of gravitation.

At first time the local gauge invariance principle was applied in order to build the gravitation theory by Utiyama in Ref.[8] by considering the Lorentz group as gauge group corresponding to gravitational interaction. Utiyama introduced the Lorentz gauge field, which has transformation properties of anholonomic Lorentz connection. By identifying this field with anholonomic connection of riemannian space-time, Utiyama obtained Einsteinian theory of gravitation by this way. The work by Utiyama [8] was criticized by many authors. At first of all, if anholonomic Lorentz connection is considered as independent gauge field, it can be identified with a connection of Riemann-Cartan continuum with torsion, but not riemannian connection [9-11]. Moreover, if a source of gravitational field includes the energy-momentum tensor of gravitating matter, we can not consider the Lorentz group as gauge group corresponding to gravitational interaction. Note that metric theories of gravitation in 4-dimensional pseudo-riemannian space-time including GR, in the frame of which the energy-momentum tensor is a source of gravitational field, can be introduced in the frame of

gauge approach by the localization of 4-parametric translation group [12, 13]¹. By localizing 4-translations and introducing gauge field as symmetric tensor field of second order, the structure of initial flat space-time changes, and gauge field becomes to connected with metric tensor of physical space-time. Because the localized translation group leads us to general coordinate transformations, from this point of view the general covariance of GR plays the dynamical role. At the same time the Lorentz group (group of tetrad Lorentz transformations) in GR and other metric theories of gravitation does not play any dynamical role from the point of view of gauge approach, because corresponding Noether's invariant in these theories is identically equal to zero [14]. The other treatment to localization of translation group was presented in [15, 16], where gravitation field was introduced as tetrad field in 4-dimensional space-time with absolute parallelism. This theory is not covariant with respect to localized tetrad Lorentz transformations, and in fact it is intermediate step to gravitation theory with independent gauge Lorentz field. If one means that the Lorentz group plays the dynamical role in the gauge field theory and the Lorentz gauge field exists in the nature, in this case we obtain with necessity the gravitation theory in the Riemann-Cartan space-time (see, for example, [17-19]). Corresponding theory is known as Poincare gauge theory of gravitation (PGTG). Gravitational field variables in PGTG are the tetrad $h^i{}_\mu$ (translational gauge field) and the Lorentz connection $A^{ik}{}_\mu$ (Lorentz gauge field); corresponding field strengths are the torsion tensor $S^i{}_{\mu\nu}$ and the curvature tensor $F^{ik}{}_{\mu\nu}$ defined as

$$S^i{}_{\mu\nu} = \partial_{[\nu} h^i{}_{\mu]} - h_{k[\mu} A^i{}_{\nu]}{}^k,$$

$$F^{ik}{}_{\mu\nu} = 2\partial_{[\mu} A^{ik}{}_{\nu]} + 2A^{il}{}_{[\mu} A^k{}_{\nu]}{}^l,$$

where holonomic and anholonomic space-time coordinates are denoted by means of greek and latin indices respectively. As sources of gravitational field in PGTG are energy-momentum and spin tensors. The simplest PGTG is the Einstein-Cartan theory based on gravitational Lagrangian in the form of scalar curvature of Riemann-Cartan space-time [10,11,23]. In certain sense the Einstein-Cartan theory of gravitation is degenerate gauge theory, in the frame of which the torsion is connected linearly with spin tensor of gravitating matter and in the case of spinless matter the torsion vanishes, although the torsion is a gravitational field strength corresponding to localized translation group. Like gauge Yang-Mills fields, gravitational Lagrangian of PGTG has to include invariants quadratic in gravitational field strengths - curvature and torsion tensors. The including of linear in curvature term (scalar curvature) to gravitational Lagrangian is necessary to satisfy the correspondence principle with GR.

We will consider the PGTG with gravitational Lagrangian given in general form contain-

¹Because in the frame of gauge approach the gravitational interaction is connected with space-time transformations, the gauge treatment to gravitation has essential differences in comparison with Yang-Mills fields connected with internal symmetries groups. As a result, there are different gauge treatments to gravitational interaction not detailed in this paper.

ing different invariants quadratic in the curvature and torsion tensors

$$\mathcal{L}_G = f_0 F + F^{\alpha\beta\mu\nu} (f_1 F_{\alpha\beta\mu\nu} + f_2 F_{\alpha\mu\beta\nu} + f_3 F_{\mu\nu\alpha\beta}) + F^{\mu\nu} (f_4 F_{\mu\nu} + f_5 F_{\nu\mu}) + f_6 F^2 + S^{\alpha\mu\nu} (a_1 S_{\alpha\mu\nu} + a_2 S_{\nu\mu\alpha}) + a_3 S^\alpha_{\mu\alpha} S^\mu_{\beta}{}^{\beta}, \quad (1)$$

where $F_{\mu\nu} = F^\alpha_{\mu\alpha\nu}$, $F = F^\mu_\mu$, f_i ($i = 1, 2, \dots, 6$), a_k ($k = 1, 2, 3$) are indefinite parameters, $f_0 = (16\pi G)^{-1}$, G is Newton's gravitational constant.

3 Regular cosmology in Riemann-Cartan space-time

According to observational data concerning anisotropy of relic radiation, our Universe was homogeneous and isotropic beginning from initial stages of cosmological expansion. In connection with this fact, the investigation of HICM is of greatest interest for relativistic cosmology. In the frame of PGTG homogeneous isotropic models are described in general case by means of three functions of time: the scale factor of Robertson-Walker metrics $R(t)$ and two torsion functions $S(t)$ and $\tilde{S}(t)$ determining the following components of torsion tensor (with holonomic indices) [20]: $S^1_{10} = S^2_{20} = S^3_{30} = S(t)$, $S_{123} = S_{231} = S_{312} = \tilde{S}(t) \frac{R^3 r^2}{\sqrt{1-kr^2}} \sin \theta$, where spatial spherical coordinates are used. The functions S and \tilde{S} have different properties with respect to transformations of spatial inversions, namely, the function $\tilde{S}(t)$ has pseudoscalar character. In the case $\tilde{S}(t) = 0$ HICM in the frame of PGTG were built and investigated in a number of papers (see [20, 5-7] and references herein).² The curvature tensor in this case has the following non-vanishing components: $F^{01}_{01} = F^{02}_{02} = F^{03}_{03} \equiv A$ and $F^{12}_{12} = F^{13}_{13} = F^{23}_{23} \equiv B$ with

$$A = \frac{(\dot{R} - 2RS)}{R}, \quad B = \frac{k + (\dot{R} - 2RS)^2}{R^2}, \quad (2)$$

and Bianchi identities in this case are reduced to the only relation

$$\dot{B} + 2H(B - A) + 4AS = 0, \quad (3)$$

where $H = \frac{\dot{R}}{R}$ is the Hubble parameter, and a dot denotes differentiation with respect to time.

The system of gravitational equations of PGTG with gravitational Lagrangian (1) is reduced to three equations, which by using (3) can be written in the following form [20]

$$\begin{aligned} 6f_0 B - 12f(A^2 - B^2) - 3a(H - S)S &= \rho, \\ 2f_0(2A + B) + 4f(A^2 - B^2) - a(\dot{S} + HS - S^2) &= -p, \\ f(\dot{A} + \dot{B}) + [f_0 + \frac{1}{8}a + 4f(A + B)]S &= 0. \end{aligned} \quad (4)$$

²Possible role of the torsion function \tilde{S} is discussed in the talk [21] of this Conference.

where $f = f_1 + \frac{1}{2}f_2 + f_3 + f_4 + f_5 + 3f_6$, $a = 2a_1 + a_2 + 3a_3$, ρ is the energy density, p is the pressure and the average of spin distribution of gravitating matter is supposed to be equal to zero. The system of equations (4) leads to cosmological equations without high derivatives if $a = 0$ [20] (see below). Then we find from (4) the curvature functions A and B and the torsion S in the following form

$$\begin{aligned} A &= -\frac{1}{12f_0} \frac{\rho + 3p - \alpha(\rho - 3p)^2/2}{1 + \alpha(\rho - 3p)}, \\ B &= \frac{1}{6f_0} \frac{\rho + \alpha(\rho - 3p)^2/4}{1 + \alpha(\rho - 3p)}, \\ S(t) &= -\frac{1}{4} \frac{d}{dt} \ln |1 + \alpha(\rho - 3p)|, \end{aligned} \quad (5)$$

where indefinite parameter $\alpha = \frac{f}{3f_0^2}$ has inverse dimension of energy density. By using expressions (2) of curvature functions for homogeneous isotropic gravitating models and the solution (5) of gravitational equations of PGTG we obtain the following generalized cosmological Friedmann equations (GCFE)

$$\frac{k}{R^2} + \left\{ \frac{d}{dt} \ln \left[R \sqrt{|1 + \alpha(\rho - 3p)|} \right] \right\}^2 = \frac{8\pi G}{3} \frac{\rho + \frac{\alpha}{4}(\rho - 3p)^2}{1 + \alpha(\rho - 3p)}, \quad (6)$$

$$R^{-1} \frac{d}{dt} \left[\frac{dR}{dt} + R \frac{d}{dt} \left(\ln \sqrt{|1 + \alpha(\rho - 3p)|} \right) \right] = -\frac{4\pi G}{3} \frac{\rho + 3p - \frac{\alpha}{2}(\rho - 3p)^2}{1 + \alpha(\rho - 3p)}. \quad (7)$$

The conservation law in PGTG has usual form

$$\dot{\rho} + 3H(\rho + p) = 0. \quad (8)$$

By using the GCFE we can investigate HICM, if the content of gravitating matter is known. So, in the case of HICM filled with non-interacting scalar field ϕ minimally coupled with gravitation and gravitating matter with equation of state in general form $p_m = p_m(\rho_m)$, the energy density ρ and the pressure p take the form

$$\rho = \frac{1}{2}\dot{\phi}^2 + V + \rho_m \quad (\rho > 0), \quad p = \frac{1}{2}\dot{\phi}^2 - V + p_m, \quad (9)$$

where $V = V(\phi)$ is a scalar field potential. By using (9) and the scalar field equation in homogeneous isotropic space

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{\partial V}{\partial \phi} \quad (10)$$

we transform the GCFE (6)-(7) to the following form [7]

$$\begin{aligned} &\left\{ H \left[Z + 3\alpha \left(\dot{\phi}^2 + \frac{1}{2}Y \right) \right] + 3\alpha \frac{\partial V}{\partial \phi} \dot{\phi} \right\}^2 + \frac{k}{R^2} Z^2 \\ &= \frac{8\pi G}{3} \left[\rho_m + \frac{1}{2}\dot{\phi}^2 + V + \frac{1}{4}\alpha \left(4V - \dot{\phi}^2 + \rho_m - 3p_m \right)^2 \right] Z, \end{aligned} \quad (11)$$

$$\begin{aligned}
& \dot{H} \left[Z + 3\alpha \left(\dot{\phi}^2 + \frac{1}{2}Y \right) \right] + 3H^2 \left[Z - \alpha\dot{\phi}^2 + \alpha Y \right. \\
& \quad \left. - \frac{3\alpha}{2} \left(\frac{dp_m}{d\rho_m} Y + 3(\rho_m + p_m)^2 \frac{d^2 p_m}{d\rho_m^2} \right) \right] + 3\alpha \left[\frac{\partial^2 V}{\partial \phi^2} \dot{\phi}^2 - \left(\frac{\partial V}{\partial \phi} \right)^2 \right] \\
& = 8\pi G \left[V + \frac{1}{2}(\rho_m - p_m) + \frac{1}{4}\alpha \left(4V - \dot{\phi}^2 + \rho_m - 3p_m \right)^2 \right] - \frac{2k}{R^2} Z. \quad (12)
\end{aligned}$$

By using (10)-(12) inflationary cosmological models were investigated in [6].

The principal difference of (6)–(7) from Friedmannian cosmological equations of GR is connected with terms containing the parameter α . These terms arise from quadratic in the curvature tensor part of gravitational Lagrangian, which unlike metric theories of gravitation does not lead to high derivatives in cosmological equations. The value of $|\alpha|^{-1}$ determines the scale of extremely high energy densities. Solutions of GCFE (6)–(7) coincide practically with corresponding solutions of GR, if the energy density is small $|\alpha(\rho - 3p)| \ll 1$ ($p \neq \frac{1}{3}\rho$). The difference between GR and PG TG can be essential at extremely high energy densities $|\alpha(\rho - 3p)| \gtrsim 1$. Ultrarelativistic matter ($p = \frac{1}{3}\rho$) and gravitating vacuum ($p = -\rho$) with constant energy density are two exceptional systems, because GCFE (6)–(7) are identical to Friedmannian cosmological equations of GR in these cases independently on values of energy density, and the torsion function vanishes. Properties of solutions of equations (6)–(7) at extreme conditions depend on the sign of parameter α and certain restriction on equation of state of gravitating matter. The study of inflationary models including scalar fields shows that GCFE (6)–(7) lead to acceptable restriction for scalar field variables if $\alpha > 0$ [5]. In the case $\alpha > 0$ all cosmological solutions have regular bouncing character, if at extreme conditions $p > \frac{1}{3}\rho$. There are physical reasons to assume, that the restriction $p > \frac{1}{3}\rho$ is valid for gravitating matter at extreme conditions [22]. Note, that this condition is valid for so-called stiff equation of state $p = \rho$ used in the theory of the early Universe (Ya. B. Zeldovich and others).

The GCFE lead to restrictions on admissible values of energy density. In fact, if the energy density ρ is positive and $\alpha > 0$, from equation (6) in the case $k = +1, 0$ follows the relation:

$$Z \equiv 1 + \alpha(\rho - 3p) \geq 0. \quad (13)$$

The condition (13) is valid not only for closed and flat models, but also for cosmological models of open type ($k = -1$) [5]. In the case of models filled with usual gravitating matter without scalar fields the equation $Z = 0$ determines limiting (maximum) energy density, and regular transition from compression to expansion (bounce) takes place for all cosmological solutions by reaching limiting energy density. In the case of systems including also scalar fields a bounce takes place in points of so-called "bounce surfaces" in space of variables $(\phi, \dot{\phi}, \rho_m)$ [5]. Near bounce surfaces as well as bounds $Z = 0$ gravitational interaction has the character of repulsion, but not attraction. The domain of energy densities and other variables of gravitating matter, when gravitational repulsion effect takes place, depends on matter properties at extreme conditions (equation of state of gravitating matter, scalar field

potentials) [7].

As it was noted above, gravitational repulsion effect ensures regular behaviour of cosmological solutions in metrics, Hubble parameter, its time derivative in the frame of classical field-theoretical description without quantum gravitational corrections. Any cosmological solution has bouncing character and includes the regular transition from compression to cosmological expansion. As illustration, below particular cosmological solution for flat inflationary cosmological model is given (Fig.1 - Fig.3) [5]. The model is filled with scalar field with potential $V = \frac{1}{2}m^2\phi^2$ ($m = 10^{-6}M_p$, M_p is the Planckian mass) and ultrarelativistic matter. The solution was obtained by numerical integration of Eqs. (10), (12) and by choosing some initial conditions at a bounce; the value of parameter $\alpha = 10^{14}M_p^{-4}$, Planckian system of units is used in Fig.1 - Fig.3. Note that duration of the stage of transition from compression to expansion is extremely small (several order smaller than duration of inflationary stage). This means, that such models correspond to regular Big Bang or Big Bounce. As numerical analysis of inflationary cosmological models in PGTG shows [6], quantitative differences of such models in comparison with similar models in GR are possible at the end of inflationary stage, if the value of $|\alpha|^{-1}$ is much less than the Planckian energy density. In this case corresponding corrections concerning anisotropy of relic radiation in considered theory are possible.

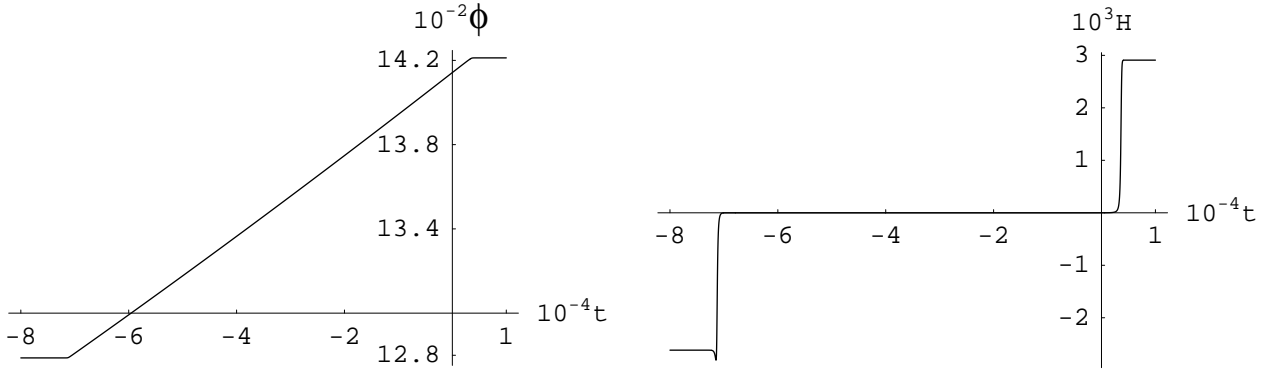


Figure 1: The stage of transition from compression to expansion.

4 Conclusion

The analysis of HICM carried out in the frame of PGTG shows, that this theory leads to the solution of principal problem of GR – problem of cosmological singularity – and permits to build regular cosmology. It is because the gravitational interaction at extreme conditions in PGTG has the character of repulsion but not attraction. This effect is connected with geometrical structure of physical space-time in PGTG, namely with the space-time torsion.

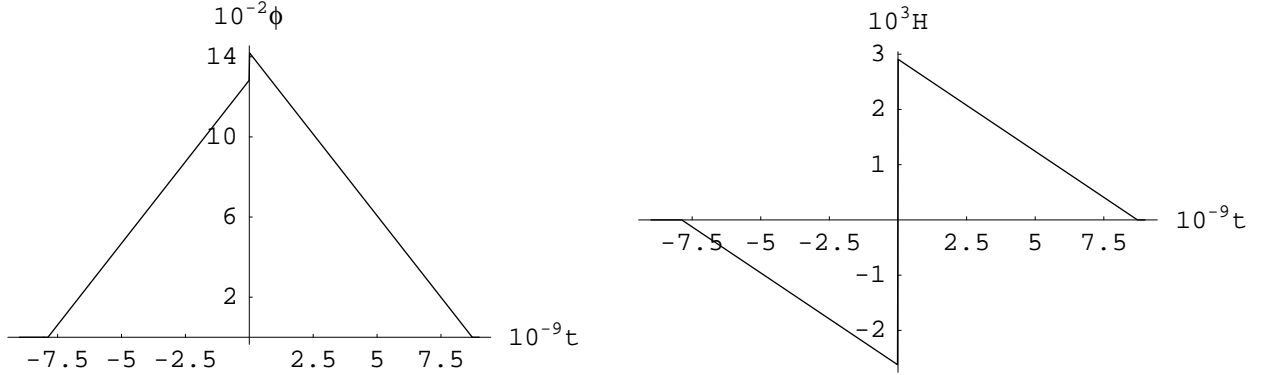


Figure 2: Quasi-de-Sitter stage of compression and inflationary stage.

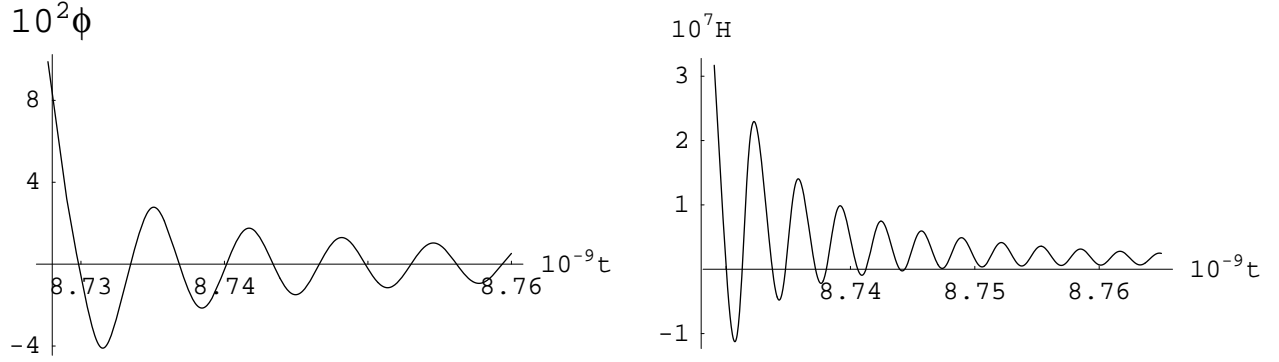


Figure 3: The stage after inflation.

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